Final Project

Will introducing mathematical reasoning in small doses in conversations and writing activities help students to expand their mathematical thinking/reasoning processes?

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1. Statement of Question:

Will introducing mathematical reasoning in small doses in conversations and writing activities help students to expand their mathematical thinking/reasoning processes?

This question is important to me because I know that my students can use mathematical reasoning, but it is "different" then what they are use to, thus I did not want them to feel overwhelmed when starting to use mathematical reasoning. I began this research project trying to think of how I can integrate math reasoning into my lessons and activities without creating a shock to the students. I teach 8th grade though 12th grade math classes, all of which include students that have never used or had very little exposure to math reasoning. By the time students have me as a teacher they have become accustomed to the teacher being the source of what is right and wrong, and the giver of procedures and formulas to find the correct answer. I did not want to have my students put up a blocker if I introduced too much math reasoning at one time. I did not want them to feel overwhelmed or feel as though math reasoning wasn't something they could handle. They are all very capable of using math reasoning, but are "afraid" of new approaches.

To help students be able to better handle being introduced to math reasoning, I felt it would be best to begin using "thinking" questions that students would have to work through individually, write about or discuss as a class. Using these approaches allowed students to begin to reason through their mathematical processes in different ways.

2. Review of Literature:

As I am beginning to get an idea of how to start using math reasoning in the classroom, I realize an important aspect of my definition of mathematical reasoning revolves around the ability of the class, students, and the teacher to be a part of genuine mathematics conversations. These conversations need to be guided by the teacher when interacting with one student, a small group of students or the entire class. How are these conversations guided in such a way that students will gain necessary concepts from them? Another important aspect in guiding students to use mathematical reasoning is to integrate independent thinking activities, such activities could be students reasoning through a new problem on their own or writing about their thinking.

Teaching with Conversations: Beginnings and Endings helped me to visualize two different whole class conversations. The examples that were given, one involving a 10^{th} grade classroom and the other an 11^{th} grade classroom, began and ended differently. These examples allowed me to further analyze how conversations can begin and end without creating frustrations or boredom among the students. It is important that conversations create "genuine student involvement, which means that students articulate their own ideas rather than produce what the teacher wants" (Brodie, 2007, p. 17). Good mathematical conversations require "participants to listen carefully to each other, to appreciate the import of others' contributions, to build on or critique others' ideas in sensitive and helpful ways, and to express their own ideas so that others can engage with them;" they also include "reasoning about mathematical ideas beyond the procedures that draw on those ideas" (Brodie, 2007, p. 17). This article focuses on how conversations can begin and end in the classroom. In the 10^{th} grade example, a student began the conversation asking if zero should be positive or negative when multiplying -2 x 0. At first, the teacher does not focus on this question, but later allows the question to develop into a conversation about the properties of zero. Allowing the learner to create the focus of a conversation is "likely to engage learners' interest, particularly if they share the question" (Brodie, 2007, p. 22). Although the teacher may not be expecting the question, it is important to give the learner a chance to explain their question and allow others to help, which may give time for the teacher and learners to see the importance of the question (Brodie, 2007). As in the 10^{th} grade example, once the teacher allowed the conversation to develop a bit, he was able to realize the importance of the concept to all of the students. As I am integrating mathematical reasoning and conversations into the classroom, it will be important for me to deviate from my "planned" objectives if and when a question arises from the students that could create an important conversation for the entire class. But, at the same time, it is just as important to not lose focus on the learning goals and allow the students to get on a "tangent" that would not be a beneficial conversation.

Just as important as the beginning of a conversation, ending the conversation on a clear, focused result will help the students to not be frustrated from the conversation or the derived results from it. This is not always easy, as both examples in the article there was (a) student(s) that were frustrated with the end result of the conversation. Brodie states "pulling a conversation together entails taking account of the diversity of ideas that have been expressed, putting them into a relationship with each other and bringing some resolution" (2007, p. 22). This is no easy task. Teachers shouldn't just tell students the conclusion to their conversation, instead they need to listen carefully throughout the conversation and relate the comments made while adding any missing information (Brodie, 2007). To ensure that students gain the most from mathematical reasoning conversations in the classroom, I need to stay focused on student contributions and how I can use their contributions, and any missing information, to tie up any loose ends.

In chapter 4 of *Teaching Problems and the Problems of Teaching*, Lambert discusses how she guides students in the beginning of the year to be comfortable with mathematical conversations. As she is going through the materials students will use in the math lessons and why they are using markers instead of pens or pencils, Lambert begins a discussion on the meaning of revision. It is a topic that many of the students are comfortable discussing and allows her to guide the students through a typical math conversation (even if they were not talking about math processes). This allowed the students to learn how to share an idea, learn from others ideas and add to or revise their own or others thinking. Although I am past the first day of school, I think that I can learn from her guidance of the conversation to help my students to begin to reason mathematically. Later in that first week of school, Lambert began having mathematical conversations, leading the students to become comfortable with the new environment and discussions they would be having in math class. While having a discussion on adding numbers and conjecturing about different additions between two given additions, Lambert is able to teach the students not only about the new activities they would be doing, but also about how to correctly (and politely) converse with each other. Lambert taught the students about “conditions,” “conjectures” and “revising conjectures” (2001, p. 66) during this conversation. The purpose of this lesson was to have them work through a conversation while teaching them “a method of studying that would make it possible for them to learn both mathematical content and mathematical practices” (Lambert, 2001, p. 66).

Dionne Cross conducted a study on mathematical argumentation and writing in the classroom. In her research during the study she found literary sources stating “both cognitive and metacognitive abilities are crucial to improvement in problem solving ability and developing mathematical expertise” (2009, p. 907). Writing is “an activity that helps students generate and connect their thoughts and ideas and consolidate their thinking” (Cross, 2009, p. 907). Writing therefore is a “valuable learning activity and has enormous potential for promoting metacognitive thinking thereby improving understanding of mathematical concepts” (Cross, 2009, p. 907). Classroom discussions also help develop student’s mathematical expertise. Discussions in which students are “able to make worthwhile contributions, ask questions, have their ideas evaluated, and receive immediate feedback are considered one of the more effective strategies for knowledge construction” (Cross, 2009, p. 908). While students are participating in these writing activities and discussions, teachers should take on a ‘facilitator’ role, not one of ‘transmitter of knowledge’; thus allowing students to be fully engaged in mathematical argumentation and writing while being guided by the teacher (Cross, 2009). From the results of Cross' research, she was able to find that that "engaging in activities reflective of both cognitive and socio-cultural views of knowledge and learning does lead to increased understanding and achievement" however, to improve student understanding even more, it be important to have created classroom norms in which students are able to know how to communicate effectively (2009, p. 927).

Being able to communicate about mathematics is an important learning tool. If fact, the National Council of Teachers of Mathematics has deemed it so important, communication is a standard for all students grades Pre-Kindergarten through Twelfth grade. Students need to be able to communicate about their mathematics orally and in writing. "When students are challenged to think and reason about mathematics and to communicate the results of their thinking to others orally or in writing, they learn to be clear and convincing" (NCTM, 2000, p. 59). By using communication activities in the classroom, teachers give students opportunities to develop their own understandings by listening to others' explanations and exploring ideas from multiple perspectives (NCTM, 2000). "Students who are involved in discussions in which they justify solutions—especially in the face of disagreement—will gain better mathematical understanding as they work to convince their peers about differing points of view" (NCTM, 2000, p. 59).

These readings helped me formulate my research question. Originally I began with the research question: "How can I begin using mathematical reasoning in the secondary classroom in a way that will help students to expand their thinking processes? Also, how can I help students to become comfortable using their math reasoning collaboratively or independently?" I hypothesized that integrating math reasoning into the classroom in small doses would best help students become comfortable using the math reasoning. From the readings that I found, I shaped my question to focus on using conversations and writing activities. This would thus help me to research how to best integrate math reasoning into the classroom while using known methods that work whether students are engaged in these activities collaboratively or independently. Using writing and conversations, focuses my research on important classroom NCTM standards.

Unfortunately, I did find limitations to the readings I consulted. I did not find any articles on how best to integrate math reasoning into classroom in which math reasoning is foreign. Although I felt it would be best to integrate the students slowly into math reasoning by using “small doses” of activities, I did not find research to support my hypothesis. However, Cross did state in her results that establishing classroom norms would have made greater improvements in student understanding (2009, p. 927). Even though this wasn’t exactly what I was looking for, knowing that establishing classroom norms for conversations and writing would improve their mathematical reasoning, it is something that I will establish in the future. Since I performed my research mid-year, I only verbally discussed some basic classroom norms with the students. Things I tried to establish were:

1. To write down anything you are thinking, right or wrong.
2. Any contribution is important in a conversation. Thus, it is important that we are polite towards others when they are contributing.

Although these “norms” were very informally given while we were participating in the discussions and activities, I hope that it helped students to feel more comfortable with the math reasoning and sharing their reasoning.

3. Modes of Inquiry:

To answer my research question, I decided to integrate conversations and writing into the regular daily schedule. Usually, new concepts would be taught with me in the front of the classroom going through the procedures needed for the new concept. I would go over sample problems together as a class then allow them a few to try on their own while I gave individual help to some students. We would then go over the problems as a class, with either volunteers or called upon students to aid in solving the problem. Instead to help facilitate my research question, I used “thinking” or guiding questions to help the students work through the process on their own or as a class. Students were asked to think through their ideas, write down their ideas and discuss their ideas with the class.

I felt this was the best method to show how integrating math reasoning in small doses to the classroom would help them to begin to use math reasoning. I had originally planned on gradually increasing the math reasoning that they would be using by including a small group activity and/or a lesson in which the students would only use math reasoning, and very little help from myself to solve a problem or situation. Time, unfortunately, was a major factor on why I changed my original plan. November is a month in which we do not have a single full week of instruction. Between professional development, parent-teacher conferences, Fall Holiday, and Thanksgiving, every week was either a 3 or 4 day week (most of them 3 day weeks). I did not see that it would be feasible to be able to get through my original plan, while still ensuring that I was covering the curriculum needed. Thus, I decided to integrate the students using math reasoning through communication and writing during the normal “lesson” time, ensuring that I could begin to get students comfortable with math reasoning while still staying on track with my curriculum.

4. Results:

I collected data for my research question on from November 17, 2010 through November 30, 2010. The data was collected from students in my Algebra 2 course. It is a very small class, consisting of only 12 students. There are five 9th graders, two 10th graders, three 11th graders and two 12th graders. The ninth graders were chosen by me last year in Algebra 1 as "advanced" students; I gave them the option of taking both Geometry and Algebra 2 in their ninth grade year to allow them to be able to take college level courses their senior year through a local community college. The tenth and eleventh graders are coming from Geometry, while the twelfth graders were in Algebra 2A last year. The twelfth graders should have followed up Algebra 2A with Algebra 2B, but it did not fit into their schedule, thus putting them into Algebra 2 to receive the needed credits for graduation. This is a 49 minute class period in the morning.

I will discuss three of the days that I collected data. On the first day of integrating math reasoning, November 17, 2010, into the classroom, we were discussing the idea of complex numbers. The next day I collected data was on November 23, 2010 in which we discussed the discriminant. On the last day of my data collection, November 30, 2010, we discussed polynomial long division. I will discuss key scenarios in which I felt the students were showing math reasoning and some cases where they were struggling with using their math reasoning.

I did face some difficulties while I was collecting data. I had planned on videotaping the lessons in which I was collecting data. Unfortunately, the hour that I chose to collect data in was the same hour as the broadcasting class. They were in the middle of a big project and needed the school cameras, thus not allowing me to videotape the lessons. Fortunately, I have my prep period following this class and that allowed me to take notes and write down things students said before too much of it was forgotten. It unfortunately was not a perfect method, and it was difficult to remember all of the individual student comments made throughout the hour.

November 17, 2010 was the first day of data collection. We were on the topic of quadratic functions and their solutions. We had already graphed quadratic functions and solved by factoring to get two real number solutions. The day before we began to discuss how a quadratic function does not always cross the x-axis, therefore it would not have any real solutions. Instead of two real solutions those quadratic functions have two complex solutions. We also discussed basic properties of complex numbers, like adding, subtracting and finding the additive inverse. On the day of data collection, I wanted to discuss multiplying complex numbers. Particularly I focused on students using math reasoning to figure out what i^2, i^3, i^4 and therefore applying the knowledge of those powers to *i* to any power.

First I put up on the board, refreshing from the definition yesterday, that $i = \sqrt{-1}$. I then asked the students to try to figure out what i^2 equals. I asked them to write down and work through what they think it might equal. I decided it would be best to just walk around the classroom to look over their thinking, but not to intervene right now; hopefully just allowing the students to think individually would help them to begin using their reasoning skills. After I noticed students were done thinking through the question I called on individual students to give their answer and explain their thinking. I did not tell them if they were right or wrong, we just worked through their thinking to come to a decision as a class. Here are a couple of the responses and our discussion:

Kelsey: I think it is either 0 or 1, but I am thinking it is probably 1

Me: Can you tell me what you did to make you think it is 1?

Kelsey: Well, $\sqrt{-1^2}$ must be 1 because -1 times -1 is positive 1 and the square root of positive one is 1.

(I wrote her thinking out on the board for the others to look at and reason through)

Me: Anyone else?

Jake: I got the same answer. But I was thinking it is because $i^2 = i \cdot i = \sqrt{-1}\sqrt{-1} = \sqrt{-1 \cdot -1} = \sqrt{1} = 1$

Me: (after I wrote it on the board) Is this the same or different than Kelsey's thinking?

Jake: Um.. it looks kinda the same.

Me: Anyone else?

Dasha: I think it is -1.

Me: Can you explain what you did to get -1?

Dasha: Well, I was thinking about $\sqrt{9} = 3$ instead and that made me think it was -1.

Me: I am a little confused, can you explain more?

Dasha: $\sqrt{9} = \sqrt{3^2} = 3$ the square root gets rid of the square like when we were solving for x the other day in the problem x^2= 9

Me: So since squaring and square-rooting are "opposite" operations they just cancel each other off so the answer is -1?

Dasha: yes.

Me: Does anyone else have any thoughts on what Dasha said?

(a few kids in class): yeah that makes sense… it's like what we did before.

Me: So $i^2 = \sqrt{-1^2} = -1$. Good job. Now let's use this thinking to find i^3.

I wasn't surprised that many of the students thought $i^2 = 1$. It is a very common misconception because it seems very logical and it uses what they know "squared" means. I was very impressed by Dasha's thinking and her ability to use what we had been doing previously, solving for x, and apply it to a very confusing topic for many students. As I allowed the students to work on figuring out i^3, this time I did stop and help a couple individual students. Here was one of my discussions:

Me: I like how you wrote out $i^3 = \sqrt{-1}\sqrt{-1}\sqrt{-1}$ Brice. Is there anything that you notice that you already know?

Brice: Yeah, we just said two of the square root of i's are equal to -1.

Me: Can you circle what you were looking at? (she circles the first two) And write what you we just discussed that means under your circle. (she did) Is there anything we still have left?

Brice: Yeah, another square root of -1. (she writes it next to the -1)

Me: Now looking at what you just wrote, is there anything that we already know?

Brice: (thinks for a bit) isn't the $\sqrt{-1} = i$?

Me: Yes it is. So now rewrite what you just noticed.

Brice: (writes $-1 \cdot i$) oh! it must be –i!

Me: Good job.

The class had similar class discussions for i^4. After we were able to get through what i^2 *and* i^3 was, I had a feeling that i^4 would be quickly figured out, which I was correct. I think that the students did a good job handling having to discuss and use math reasoning for the first time. Although there were some students that did not write anything down at the beginning, and some that erased their "wrong" answers in their notes after we held a class discussion, I think as a whole the class was able to start getting comfortable thinking and talking about their thinking.

Another day I collected data was on Tuesday, November 23rd. We were still on the topic of quadratic functions and solving them. The day before we began to use the quadratic formula to find out the solutions of a quadratic function instead of using factoring. We can find out how many spots a quadratic equation crosses the x-axis by using the discriminant of the quadratic formula. This lesson was about investigating the discriminant.

I first asked the students what a graph would look like for a function $ax^2 + bx + c = 0$. I told them they had to write down something, doesn't matter if it is right or wrong, and they can't erase anything they write down today. I was surprised at how many students were not able to remember what a quadratic function graph looked like. (We just graphed these 3 weeks ago!). I had to reinforce as I was walking around the room that they had to write down something, many students were staring blankly and didn't want to write down the "wrong" thing. After I was sure everyone wrote down something, I called on a volunteer to tell the class what the graph looked like.

Next I asked "When we solve for x, what information does that tell us about the graph?" Again, I had the students think about it and write it down. This was something that we had already discussed when we were solving by factoring. Again, I was surprised at how many students did not know this. I had a student volunteer that when we solve for x it tells us if it opens up or down. I asked how do we know by looking at the equation how it opens, another student responded by stating that the *a* tells us if it opens up or down and I received a lot of "oh yeah's." I would have liked to go into more of a discussion (maybe having them actually graph a quadratic function and factor it to solve) on this, but the hour was quickly disappearing. I then drew three quadrant planes on the board and told them that the graph could look three different ways, giving us different kinds of answers when we solve. I asked them if they had an idea what the three graphs would look like. They stared at me blankly. I think it was how I asked the question, so I said I would draw the first possibility for them. I drew a quadratic function that crossed the x-axis at two spots and asked them how many solutions this graph would have. They all said two. A student then offered that the U could just cross the x-axis once, having one solution. I drew that on the board. Another student stated that the U could not cross the x-axis. Someone else said that there would be no solutions, since it does not cross the x-axis. I told them that the problems that had imaginary solutions do not cross the x-axis.

I then had them all write down the quadratic formula and use their assignment from last night to try to figure out which part of our quadratic formula could be used to see if a quadratic function has two, one or no real solutions. As I walked around I asked them to circle the part that they thought would give us that information. I saw many students circling the discriminant and some circling other parts.

Me: Which part should I circle that would tell us how many solutions the quadratic function has?

Alyssa: Circle the square root part.

Me: Can you tell me why the square root part will tell us how many solutions it has?

Alyssa: Because in the problems yesterday that was how we got the i part of our answer.

Me: Good. (Writing the discrimant on the board, then to class) what kind of number would we get under the square root to have two real answers?

Student: Perfect square numbers.

Me: What do you mean?

Student: Numbers you can take the square root of.

Me: Like, for example, $\sqrt{9}$?

Student: Yeah.

Me: Would any other numbers give us two real solutions? (no response) Okay, let's look at one solution then. How could we get just one solution?

Students gave ideas where the number is not a perfect square or when we can simplify “all three numbers.” I put up examples on the board of what I thought the students were talking about and after they said that was an example they were thinking of, we dissected them to find out that we would have two solutions. I then decided to go back to the two solutions and no solutions and state in new way what we had already discussed.

Me: How about we think about this another way. What kind of numbers are under the square root for two imaginary solutions?

Class: Negative.

Me: (Writing down $b^2 - 4ac < 0$) is this right? (class-yes) What about when we have two solutions?

Jacob: $b^2 - 4ac > 0$ or they are positive!

Me: Good. Now how about the one solution?

Jacob: Oh, oh! It must be equal to zero!

Me: (writing an example $\frac{-3 \pm 0}{3}$ on board) so like in this case?

Jacob: Yeah!

I think that this discussion went well, but the students really struggled with how to explain their thinking when talking about the discriminant. I also think that I may have helped too much at times, not really knowing what to say all the time to best give them an opportunity to reason it out themselves and balance the fact that we needed to get through the discussion before the end of the hour. That is the unfortunate fact of teaching; I only have a limited amount of time to get a concept across. I was very unsure of how to allow them to keep thinking through their reasoning with still being able to get through the concept before they left me for the day.

My last day of data collection was on November 30, 2010. We had since moved past quadratic functions and began to cover all polynomial functions. For this data collection we were discussing polynomial long division. I began the lesson having the students perform some basic long division and apply those procedures to dividing polynomials. During this lesson I felt the students were showing more confidence when sharing their explanations and I saw each of the students trying to work out their thinking individually when asked. I will focus on one part of the lesson in which a student did not want to do long division on such an easy problem and also asked about a different kind of problem.

> Me: Now tell me how you guys did 72 ÷ 6? How do I first set it up?
>
> Class: Put 72 under the division and 6 on the outside.
>
> Me: Okay. Abbie what do I do next?
>
> Abbie: It's 12.
>
> Me: But tell me how we can use long division show it is 12.
>
> Abbie: I don't know, I just know it is 12.
>
> Me: That is great you can do this problem in your head, but we are trying to look at how we can use the process of long division to help when we start looking at polynomials. Can anyone else explain what we do?

The class was easily able to walk me through the steps of using long division, but I could see that Abbie was still frustrated with the fact that we even had to do it that way when she knew the answer.

> Me: Does everyone see how we used long division here? Abbie?

Abbie: I don't see why we used it, 6 goes into 72 evenly, we don't need to do all that work.

Me: True, we can figure it out easily without long division, but I wanted to review on something easy.

Abbie: What if it doesn't go in evenly?

Me: Let's try one like that. Let's try $97 \div 6$. Help me through that problem.

Abbie was able to walk me through using long division in that case. When I asked her what we do with the remainder, someone stated write R 2.

Me: We did back in early elementary, how do we write remainders now?

Abbie: As a fraction, like in this case 2/6. But that would be 1/3.

Me: Good job. Keep that in mind, we will use remainders in a little bit.

To help Abbie become more comfortable using long division so she could apply the process with the polynomials, I felt it was important to spend time on her example, even though I had a remainder problem planned a little later in the lesson. It was important to her that we discuss it when the question arose, not when I had planned to get to it.

5. Conclusions and Limitations:

With a limited amount of time to conduct my study I felt that I was not given ample opportunity to fully develop the potential of students' mathematical reasoning. By integrating thinking questions and small opportunities for students to think, write and talk about their mathematical reasoning, I feel the students began to be comfortable discussing their ideas, using their prior knowledge to learn something new and think independently about a new concept. Unfortunately, there were a few students who were not comfortable with the lessons that included math reasoning and did not want to explain their thinking. They were quiet or did not want to explain their thinking when asked. I am unsure of how to best increase those students comfort levels when using math reasoning.

There are limitations to my conclusions, as I cannot say that the students increased ability to explain their thinking was because I introduced the students to using mathematical reasoning in small doses. I was not able to conduct a comparison group in which I would introduce mathematical reasoning on a larger scale. Thus, I am not able to accurately know that integrating math reasoning in smaller doses increased the students comfort level. But, I was able to conclude that the students did gain a higher confidence level when explaining their thinking when comparing how the students participated in the beginning to the end of my study.

The literature that I reviewed supports my findings in a few ways. NCTM states that “when students are challenged to think and reason about mathematics and to communicate the results of their thinking to others orally or in writing, they learn to be clear and convincing” (NCTM, 2000, p. 59). I observed that the students were able to be more clear and convincing in their explanations in the final day of data collection than when we first began. Lampert and Cross both stressed the importance of establishing norms for the students in their discussions and writing. Informally I was able to establish some norms for the students while they were working individually or discussing as a class. This is an important aspect to ensure that the students feel comfortable explaining their thinking to others.

I feel that I had a difficult time with what I should be doing to help the students use their math reasoning. Brodie suggested that it is important for students to “articulate their own ideas rather than produce what the teacher wants” (Brodie, 2007, p. 17). This is where I think I really struggled. In such a short class period, it is difficult to not steer the students in a direction that will help the discussion come to a desired result. The students need to learn the concept before they leave my class, time is a frustrating factor. I do, however, feel that I am improving in allowing the student questions to direct the discussion. Abbie's questions and frustrations were very important, thus I knew we needed to discuss them right then and not wait until the time in the lesson I planned to cover remainders.

6. Next Steps:

As a result of this study, I have a few new questions that I would like to further study. Since I was unable to compare this class to one that would be introduced to using math reasoning in a larger dose, I would like to see if there would be any differences in the students comfort level or amount of participation in discussions. I feel that this would be best begun at the beginning of the year when I could establish classroom norms and students would not have to get accustomed to a change my teaching.

Another question that I have would be: "does integrating mathematical reasoning into the classroom would improve students understanding of the concepts?" I could investigate this by looking at informal and formal assessments. I would need to compare a class that is using mathematical reasoning to one that is not. Since most of the classes that I teach I only have one section of it, I would have to compare the assessment results to a previous year.

In conclusion I feel that this study has opened my students' abilities to use mathematical reasoning and increased my abilities to help foster their mathematical reasoning. Although I did not always know how I should best guide student thinking, I did learn many ways to help increase their reasoning skills. I hope to continue to integrate mathematical reasoning into not only my Algebra 2 class but my other classes as well.

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